Mental Computation: The Benefits of Informed Teacher Instruction

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This study investigated the change in student mental computation strategies for addition and subtraction following eight half-hour lessons over an eight-week period The principal researcher provided the teacher with a theoretical background for mental computation and support materials for the development of the instructional program Twenty-one Year 2 students participated in pre- and post-testing using individual interviews to identify the students' mental computational methods The results indicated that students who employed inefficient methods such as *counting* moved to more sophisticated strategies such as *wholistic compensation* Other students who already employed some sophisticated strategies increased their repertoire

In Queensland schools, written arithmetic procedures for addition and subtraction have traditionally been introduced at an early stage Typically, children in Queensland are expected to be able to calculate two-digit addition and subtraction written algorithms by the end of Year 2 However, the new Queensland *Mathematics Years 1 – 10 Syllabus* (Queensland Studies Authority, 2004) incorporates mental computation into the Number Strand; although, at the level appropriate for Year 2, mental computation strategies only appear to relate to solving number facts or related number facts The inclusion of mental computation in mathematics curricula has been recommended by mathematics researchers (Cobb & Merkel, 1989; Reys & Barger, 1994; Sowder, 1990; Willis, 1990); reasons for its inclusion being that mental computation (1) enables children to learn how numbers work, make decisions about procedures, and create strategies (Reys, 1985; Sowder, 1990); (2) promotes greater understanding of the structure of number and its properties (Reys, 1984); and (3) can be used as a '*vehicle* for promoting thinking, conjecturing and generalising based on conceptual understanding' (Reys & Barger, 1994, p 31) Work by Anghileri (2001) and Treffers (1998) indicated that the early introduction of formal computational procedures may well be an impediment to the development of number sense as well as cognitive and metacognitive mental computational strategies (McIntosh, Reys & Reys, 1992) For these reasons early introduction of mental computational strategies prior to the introduction of formal addition and subtraction algorithms was the focus of this study

A wide variety of mental addition and subtraction strategies has been identified in the literature (Beishuizen, 1993; Blöte, Klein & Beishuizen, 2000; Cooper, Heirdsfield & Irons, 1996; Reys, Reys, Nohda, Ishida, Yoshikawa & Shimizu, 1991; Thompson & Smith, 1999) These strategies are summarised in Table 1 In terms of efficiency, Thompson and Smith (1999) classified the strategies so that *aggregation* and *wholistic* were the most sophisticated Similarly Heirdsfield and Cooper (1997) argued that *separation right to left, separation left to right, aggregation and wholistic* represented increasing levels of strategy sophistication The terms *1010* and *u-1010* are used for *separation* strategies in the Dutch literature, *N10* and *u-N10* are used for the *aggregation* strategies, and *N10C* is used for the compensation strategy which is described here as *wholistic* (e g , Blöte, Klein & Beishuizen, 2000)

Beishuizen (1999) argued that when children's mental computational strategies are supported by the empty number line (ENL), efficient mental computation strategies are stimulated and many alternative strategies are opened to the students If students are initially exposed to jumps/leaps of ten on the ENL leading to larger jumps of multiples of ten, students will recognise shortcut strategies for addition and subtraction for numbers such as 9, 19, 11 and 21 These strategies are listed in Table 1 and are the focus of this paper Other materials that have been utilised to support the development of mental computation strategies include hundred square and arithmetic blocks, with varying degrees of success (Beishuizen, 1993)

Table 1
Mental strategies for Addition and Subtraction (based on Beishuizen, 1993; Cooper et al , 1996; Reys et al , 1995; Thompson & Smith, 1990)

Strategy		Example
Counting		28+35: 28, 29, 30, (count on by 1) 52-24: 52, 51, 50, … (count back by 1)
Separation	*Right to left (u-1010)*	28+35: 8+5=13, 20+30=50, 63 52-24: 12-4=8, 40-20=20, 28 (subtractive) : 4+8=12, 20+20=40, 28 (additive)
	Left to right (1010)	28+35: 20+30=50, 8+5=13, 63 52-24: 40-20=20, 12-4=8, 28 (subtractive) : 20+20=40, 4+8=12, 28 (additive)
	Cumulative sum or difference	28+35: 20+30=50, 50+8=58, 58+5=63 52-24: 50-20=30, 30+2=32, 32-4=28
Aggregation	*Right to left (u-N10)*	28+35: 28+5=33, 33+30=63 52-24: 52-4=48, 48-20=28 (subtractive) : 24+8=32, 32+ 20=52, 28 (additive)
	Left to right (N10)	28+35: 28+30=58, 58+5=63 52-24: 52-20=32, 32-4=28 (subtractive) : 24+20=44, 44+8=52, 28 (additive)
Wholistic	*Compensation (N10C)*	28+35: 30+35=65, 65-2=63 52-24: 52-30=22, 22+6=28(subtractive) : 24+26=50, 50+2=52, 26+2=28 (additive)
	Levelling	28+35: 30+33=63, 52-24: 58-30=28 (subtractive) : 22+28=50, 28 (additive)
Mental image of pen and paper algorithm		Child reports using the method taught in class, placing numbers under each other, as on paper, and carrying out the operation, right to left

Method

This research adopted a case study design in which a teaching experiment was conducted where a Year 2 teacher's knowledge of mental computation was developed and supported The aim was to transform her Year 2 (approximately 7 years of age) students' mental computation methods by the use of sophisticated addition and subtraction mental computation strategies as listed in Table 1 Prior to this, students had not engaged in developing mental computation

Twenty addition and subtraction word problems, incorporating 1-, 2-, and 3-digit examples were asked of each student during individual interviews prior to instruction in mental computation The stimulus pictures and numerals were presented on card to the child, while the interviewer verbalised the word problem One-digit examples (e g , 5+9) are considered number facts, but for the purposes of this paper, such examples will be discussed, as the solution strategies give interesting insights into students' computational development Such Derived Facts Strategies (DFS) (Steinberg, 1985) as *use doubles* (e g , 6+7 = double 6 plus 1) and *go through 10* (e g , 8+5 = (8+2) + 3) (Thornton, 1990) were identified in the pre-test and formed part of the teaching in the classroom This paper reports on the strategies used for 5 addition and 5 subtractions questions These questions were a direct reflection of the teaching that occurred during this teaching experiment

Instruction involved a half hour lesson on one day per week, for eight weeks The teacher was supported in the classroom by the principal researcher who provided a teaching sequence, based on previous research (Heirdsfield, 2004), feedback on lessons, and suggestions for future planning While it was acknowledged that the ENL might provide the most promise for developing mental computation strategies (Beishuizen, 1993), research conducted in the previous year (Heirdsfield, 2004) seemed to indicate that the ENL and the hundred square were both beneficial for the development of mental computation strategies Further, the children had been introduced to base 10 material in the form of bundling sticks While bundling sticks are traditionally used to support *separation* strategies *(u-1010* or *1010*), in this teaching experiment the students were encouraged to hold one number as a whole (e g , 23) and count on or back in tens (e g , 23, 33, 43, 53,) Therefore, *aggregation* strategies (*N10*) were encouraged Consequently, a variety of materials was incorporated in to the teaching sequence The following sequence for introducing number combinations in conjunction with appropriate models (number line, empty number line, hundreds chart, bundling sticks) was followed (Heirdsfield, 2004):

1. jumping in tens forwards and backwards from multiples of ten (eg start with 30 – jump forwards or backwards in tens)
2. jumping in tens forwards and backwards (eg start with 53 – jump forwards or backwards in tens)
3. relate the previous step to addition and subtraction (eg Start with 53 – add 10, add 20, add 30; take away 10, take away 20, etc)
4. further addition and subtraction without bridging tens (e g , 43 $\pm$ 21)
5. further addition and subtraction, bridging the tens (e g , 47 $\pm$ 9; 47 $\pm$ 19)

At the completion of the eight-week period, the students were again individually interviewed using the identical items as those presented in the pre-test The results from these two tests were compared for changes in strategy use and accuracy levels

Results and Discussion

In Table 2 the number of correct responses to the addition examples in the pre and post interviews is documented The students improved in accuracy on the five addition items from 28 6% correct on the pre-test to 68 6% correct on the post-test While increase in accuracy is acknowledged as being important, our focus in this paper is directly related to change in mental computation method from the pre-test to post-test

Table 2

Frequency of Correct Answers to Addition Questions (n=21)

	5+9		20+30		23+19		26+9		36+99	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Number correct	11	20	8	19	3	12	5	11	3	10

In Table 3 the number of correct responses for the subtraction examples is documented For subtraction, the students improved in accuracy on the five subtraction items from 21 0% correct on the pre-test to 47 6% correct on the post-test It is important to note that the increase in overall correct did not improve at the same rate as with addition This may be directly attributed to the reduced time spent on subtraction during the intervention classes compared to the time spent on addition

Table 3

Frequency of Correct Answers to Subtraction Questions (n=21)

	15 - 9		30 - 10		46 - 20		30 - 19		134 - 99	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Number correct	6	11	10	16	4	10	1	7	1	6

Student Mental Computation Addition Strategies – Pre- and Post-test

Table 4

Frequency of Mental Computation Strategies used for Addition (n=21)

Strategy	5 + 9		20 + 30		23 + 19		26 + 9		36 + 99	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Incorrect /inappropriate /no strategy	6	1	9	1	12	8	10	3	11	7
Counting	9	8	5	1	7	3	10	7	7	2
Derived fact strategy	6	12								
Separation										
Right to Left							1	1		
Left to Right			7	19	1	2		2		
Cumulative sum or difference					1	2				
Aggregation										
Right to Left										
Left to Right								1		
Wholistic										
Compensation						5		6	3	11
Levelling						1		1		1

In Table 4, the shift in mental computation strategy for each addition question from the pre-test to the post-test is documented It should be noted that the figures do not always denote <u>correct</u> strategy use; merely <u>strategy</u> use Pre-test strategies tended towards the least efficient strategies; while post-test methods tended towards the more efficient strategies For example, with the question 20 + 30, on the pre-test, 9 students did not use an appropriate or correct strategy and 5 used *counting*; while on the post-test, only 2 students were in these categories with 19 students using *separation L to R* It is important to note that the students who had no/incorrect strategy on the pre test tended to employ *counting* in the post-test and the students who counted in the pre-test tended to move to more sophisticated mental computation strategies in the post-test

The interview discussions on student mental computation method comparing pre- and post-test clearly indicated a shift from lower order strategies such as *counting* to more sophisticated strategies such as *wholistic compensation*

Interview Responses to 5 + 9

Molly <u>Pre</u>: (Used fingers, started at 5 and gave her answer as 13) Strategy: Count on
Molly <u>Post</u>: *Ten plus 5 equals 15, less one equals 14* Strategy: DFS

Jackson <u>Pre</u>: *59* Strategy: Incorrect strategy
Jackson <u>Post</u>: *Five plus 10 is 15, 9 is one less so it's 14* Strategy: DFS

Interview Responses to 20 + 30

Lachlan <u>Pre</u>: *Too hard* Strategy: No strategy
Lachlan <u>Post</u>: *Two plus 3 equals 5, 50* Strategy: Separation L to R

Camelia <u>Pre</u>: *90 (counted 20 on from 30)* Strategy: Count on
Camelia <u>Post</u>: *Two plus 3 equals 5, so 20 plus 30 equals 50* Strategy: Separation L to R

Interview Responses to 23 + 19

Breanna <u>Pre</u>: Began counting by 5s Not sure, too hard Strategy: Incorrect strategy
Breanna <u>Post</u>: *Twenty-three plus 20 equals 43 One less equals 42* Strategy: Wholistic compensation

Nicholas <u>Pre</u>: *Twenty plus 19 equal 39 Then put 3 on 42* Strategy: Cumulative Sum
Nicholas <u>Post</u>: *Nineteen is a 20, that equals 43 Take one away* Strategy: Wholistic compensation

Interview Responses to 26 + 9

Jackson <u>Pre</u>: *$9 26* (Guessed) Strategy: No strategy
Jackson <u>Post</u>: *Got one from the 6 and there was 5 left, 35* Strategy: Wholistic Levelling

Mitchell <u>Pre</u>: *36* (Counted on in head, starting at 26) Strategy: Count on
Mitchell <u>Post</u>: *Take one off 6 add to 9 10 plus 25 equals 35* Strategy: Wholistic Levelling

Interview Responses to 36 + 99

Mitchell <u>Pre</u>: *Number is too big Can you change it to make it easier?* Strategy: No strategy
Mitchell <u>Post</u>: *Take one off 6 that makes 100 100 plus 35 equals 135* Strategy: Wholistic Levelling

Joshua <u>Pre</u>: Counted on from 99 No answer given Strategy: Count on
Joshua <u>Post</u>: *Make 99 into 100 That's 135* Strategy Pre: Wholistic Compensation

The eight-week period of one half-hour lesson per week had a significant impact on student mental computational method, with students moving away from the inefficient *counting* strategies, as they opted for more efficient and more sophisticated methods as listed in Table 4

Student Mental Computation Substraction Strategies – Pre- and Post-test

The subtraction examples given to the students tended to be solved by *separation* It is important to note that *counting*, as seen on the pre-test was substituted with the more appropriate *separation* strategies on the post-test This change can be clearly seen in Table 5 and in the interview transcripts below

Table 5
Frequency of Mental Computation Strategies used for Subtraction (n=21)

Strategy	15 - 9		30 - 10		46 - 20		30 - 19		134 - 99	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Incorrect /inappropriate /no strategy	10	9	11	4	13	8	17	9	17	14
Counting	10	5	2	1	4		2	3	2	
Derived fact strategy	1	7								
Separation										
Right to Left								1		
Left to Right			7	15	4	12	2	5		
Cumulative sum or difference			1	1						
Aggregation										
Right to Left										
Left to Right										
Wholistic										
Compensation						1		3	2	7
Levelling										

Interview Responses to 15 – 9

Breanne Pre: *Too hard I can't do it* Strategy: No strategy
Breanne Post: *6 Counted back from 15 with fingers* Strategy: Counting

Sean Pre: *6 Used fingers to count down form 15* Strategy: Counting
Sean Post: *15 minus 10 is 5, plus 1 is 6* Strategy: DFS

Interview Responses to 30 – 10

Breanna Pre: *Too hard I can't do it* Strategy: No strategy
Breanna Post: *If you count on 10 it would be 40, so 30 take away 10 equals 20* Strategy: Separation L to R

Interview Responses to 46 – 20

Laura-Beth Pre: *Too hard Tried to count back from 46 in ones* Strategy: Counting
Laura-Beth Post: *Forty minus 20 equals 20 Plus the last number on the 40, so that's 26* Strategy: Separation L to R

Joshua Pre: *5* Strategy: No strategy
Joshua Post: *Twenty plus 20 equals 40 So 40 take away 20 is 20 plus the 6 left over* Strategy: Separation L to R

Interview Responses to 30 – 19

Lachlan Pre: *No Answer* (Tried counting backwards by ones) Strategy: Count back

Lachlan Post: *Nineteen is close to 20 Thirty minus 20 is 10 11* Strategy: Wholistic compensation

Jackson Pre: *That's too hard* Strategy: No strategy

Jackson Post: *Ten less is 20 But it's 2 tens So 19 less is 11* Strategy: Wholistic compensation

Interview Responses to 134 – 99

Molly Pre: *This is too hard* Strategy: No strategy

Molly Post: *Turn 99 into 100 and then put one on, 135* Strategy: Wholistic compensation

This eight-week period has seen a significant growth in student knowledge and use of sophisticated mental calculation strategies for addition and subtraction The ongoing support given to the class teacher ensured that she was well informed and had access to appropriate models to encourage solution methods from her students During the course of this teaching experiment it became clear that the use of the number line and the ENL were more efficient models for calculation, demonstration and communication of strategies than were the bundling sticks The bundling sticks seemed to distract the children from their calculation The hundred square was used initially for the students to investigate patterns in numbers to one hundred; for instance, numbers with the same ones lie below each other, and an efficient method for counting in tens was discussed However, the ENL showed more promise for developing mental calculation methods

Because the students had not used (or seen) an ENL before, the teacher introduced number lines with multiples of ten marked at regular intervals Such activities as counting forwards and backwards (and marking the jumps) in tens, and marking where others numbers lie between the multiples of ten, helped ease the transition to the ENL

Students were encouraged to communicate their own solution methods to the class, and the use of the number line and ENL not only supported the calculation of efficient solutions but also allowed ease in demonstration of the students' methods to the class, fostering an environment of mathematics discourse on mental computation where the gradual development of both cognitive and metacognitive strategies allowed the students to construct their own solution methods

Discussion

This teaching experiment has given fruitful insight into the potential for young students to develop and efficiently use a range of mental computation strategies However, the success of this teaching experiment depended on the teacher being informed in the use of mental calculation strategies, as well as the appropriate pedagogy

If Queensland teachers are to embrace the new mathematics syllabus then teachers need a stronger foundation of the mathematics of mental computation and the ability to use this important calculation method and efficient strategies of their own This study provides further evidence for the need for continuing professional development, as well as more focussed teacher education programs, to improve teacher content knowledge on mental computation along with pedagogy specifically focussed on the importance of this new and challenging addition to the Number strand of the new syllabus Since this thinking also underpins the development of number sense, place value and the use of the operations, this fundamental way of thinking needs to be pursued in order to give students the foundational mathematics understanding necessary for them to confidently proceed into higher mathematics

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